Apéry, Bessel, Calabi-Yau and Verrill.

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Introduction.

In [4] Bailey et al (among other things) study the Bessel moments

$$c_{m,k} = \int_0^\infty x^k K_0(x)^m dx$$

Here $K_0(x)$ is a certain Bessel function that conveniently can be defined by

$$K_0(x) = \int_0^\infty e^{-x\cosh(t)} dt$$

This leads to another representation (in Ising theory)

$$c_{m,k} = \frac{k!}{2^m} \int_0^\infty \dots \int_0^\infty \frac{dx_1 \dots dx_m}{(\cosh(x_1) + \dots + \cosh(x_m))^{k+1}}$$

(historically it was the other way around).

In J.Borwein-Salvy [5] recursion formulas for the $c_{m,k}$ are derived (m fixed). In the first section these recursions are studied in more detail. E.g. if we define

$$d_n = \frac{16^n}{n!^2} c_{4,2n+1}$$

we find an Apéry-like recursion (compare [3]) and recognize formulas from [1] and [3]. Similar transformations of $c_{5,2n+1}$ lead to a 4-th order differential equations whose mirror at $x = \infty$ is a Calabi-Yau equation found by Verrill (#34 in the "big table" [2]). This is also the case with $c_{6,2n+1}$ where the differential equation at ∞ is of order 5 (also found by Verrill) with a Calabi-Yau pullback of order 4 (#130 in [2]).

There is an infinite sequence of differential equations of Verrill where the coefficients are

$$A_n^{(m)} = \sum_{i_1 + \dots i_m = n} (\frac{n!}{i_1! \dots i_m!})^2$$

In [6] she gives a rather complicated formula for computing the recursion of $A_n^{(m)}$. In the second part we simplify this essentially using ideas in J.Borwein-Salvy [5].

In the last section we prove the

Main Theorem For $m \geqslant 3$ we have

$$y = \sum_{n=0}^\infty \frac{1}{4^n n!^2} c_{m,2n+1} x^n$$

and

$$w = \sum_{n=0}^{\infty} A_n^{(m)} x^{-(n+1)}$$

satisfy the same Picard-Fuchs differential equation of order $m_+ = m/2$ if m is even and $= (m+1)/2$ if m is odd.This equation is easily found by a Maple program.

There is a simplified version of this result for Bessel fans:

The differential equation satisfied by

$$y = \sum_{n=0}^{\infty} c_{m,2n} x^{2n}$$

also has the solution

$$w = x^{-1} I_0(x^{-1})^m$$

This depends on the identity

$$I_0(4\sqrt{x})^m = \sum_{i_1+\ldots i_m=n} \frac{1}{i_1!^2\ldots i_m!^2} x^n$$

I. Some examples.
Four Bessel Functions

On p.13 in [4] Bailey et al define

$$c_{4,2n+1} = \int_0^{\infty} x^{2n+1} K_0(x)^4 dx$$

where K_0 is a Bessel function. In [5] the following recursion is derived

$$64(k+3)c_{4,k+4} - 4(k+2)(5k^2+20k+23)c_{4,k+2} + (k+1)^5 c_{4,k} = 0$$

We make the substitution

$$d_n = \frac{16^n}{n!^2} c_{4,2n+1}$$

and get the recursion

$$(n+2)^3 d_{n+2} - 2(2n+3)(5n^2+15n+12)d_{n+1} + 64(n+1)^3 d_n = 0$$

Then

$$y = \sum_{n=0}^{\infty} d_n x^n$$

satisfies the differential equation where $\theta = x\frac{d}{dx}$

$$\theta^3 - 2x(2\theta+1)(5\theta^2+5\theta+2) + 64x^2(\theta+1)^3$$

which we recognize as equation (α) in [1]. Then

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k}$$

satisfies the recursion with initial values $A_{-1} = 0$, $A_0 = 1$. Let B_n be the solution with $B_0 = 0$, $B_1 = 1$. Then we have

Theorem. We have

$$d_n = \frac{7}{8} A_n \zeta(3) - 3B_n$$

Proof. In [4] we find $c_{4,1} = \frac{7}{8}\zeta(3)$ and $c_{4,3} = \frac{7}{32}\zeta(3) - \frac{3}{16}$ giving $d_0 = \frac{7}{8}\zeta(3)$ and $d_1 = \frac{7}{2}\zeta(3) - 3$ Then we use the recursion.

We want to find the asymptotic behaviour of A_n and d_n as $n \to \infty$. Making the Ansatz

$$A_n = Cn^b \lambda^n$$

in the recursion we find $\lambda = 16$ or $\lambda = 4$ and $b = -\frac{3}{2}\cdot$ Numerical experiments suggest

$$A_n \sim 0.36 \frac{16^n}{n^{3/2}}$$

and

$$d_n \sim 0.7 \frac{4^n}{n^{3/2}}$$

This gives

$$\frac{7}{24}\zeta(3) - \frac{B_n}{A_n} \sim \frac{C}{4^n}$$

which proves

$$\frac{B_n}{A_n} \to \frac{7}{24}\zeta(3)$$

Remark. The differential equation

$$\theta^3 - 2x(2\theta+1)(5\theta^2+5\theta+2) + 64x^2(\theta+1)^3$$

is self dual at infinity and the coefficients can be written (H.Verrill, [6])

$$A_n = \sum_{i+j+k+l=n} (\frac{n!}{i!j!k!l!})^2$$

Five Bessel functions.

Consider

$$c_{5,2n+1} = \int_0^\infty x^{2n+1} K_0(x)^5 dx$$

Then using the ideas of [5] we find the recursion

$$225c_{5,n+6}-(259n^2+1554n+2435)c_{5,n+4}$$

$$+(35n^4+280n^3+882n^2+1288n+731)c_{5,n+2}-(n+1)^6c_{5,n}=0$$

Make the substitution

$$d_n=\frac{15^{2n}}{n!^2}c_{5,2n+1}$$

which gives the recursion

$$n^2(n-1)^2d_n=4(n-1)^2(259n^2-518n+285)d_{n-1}$$

$$-3600(35n^4-210n^3+483n^2-504n+201)d_{n-2}+3240000(n-2)^4d_{n-3}$$

Let A_n be the solution of the recursion with initial values $A_0=1$, $A_1=0$, $A_2=0$. Similarly let B_n and C_n be solutions with $B_0=0$, $B_1=1$, $B_2=0$, $C_0=0$, $C_1=0$, $C_2=1$ respectively. Then

$$d_n=A_ns+225B_nt+C_n(6750-4500s+64125t)$$

where $s=c_{5,1}$ and $t=c_{5,3}$. We also use the conjectured value of $c_{5,5}=\frac{8}{15}-\frac{16}{45}s+\frac{76}{15}t$. Unfortunately we still do not know the exact values of s and t. Maybe they are related to the Apéry limits of $\frac{B_n}{A_n}$ and $\frac{C_n}{A_n}$

A related Calabi-Yau equation.

With $\theta=x\frac{d}{dx}$ the differential equation satisfied by

$$y=\sum_{n=0}^{\infty}d_nx^n$$

is

$$\theta^2(\theta-1)^2-4x\theta^2(259\theta^2+26)+3600x^2(35\theta^4+70\theta^3+63\theta^2+28\theta+5)-3240000x^3(\theta+1)^4$$

The last factor cointains $(\theta+1)^4$ which suggests that transforming the equation to $x=\infty$ could give a Calabi-Yau equation. This is indeed the case: The substitutions $\theta\longrightarrow-\theta-1$ and $x\longrightarrow 900x^{-1}$ give

$$\theta^4-x(35\theta^4+70\theta^3+63\theta^2+28\theta+5)$$

$$+x^2(\theta+1)^2(259\theta^2+518\theta+285)-225x^3(\theta+1)^2(\theta+2)^2,$$

an equation found by Helena Verrill [6] . It has #34 in the big table [2] and has the analytic solution

$$y=\sum_{n=0}^{\infty}a_nx^n$$

where

$$a_n = \sum_{i+j+k+l+m=n} (\frac{n!}{i!j!k!l!m!})^2$$

Six Bessel functions.

Consider

$$c_{6,k} = \int_0^\infty x^k K_0(x)^6 dx$$

As above we have

$$2304(k+4)c_{6,k+6} - 16(k+3)(49k^2+294k+500)c_{6,k+4}$$

$$+8(k+2)(7k^4+56k^3+182k^2+280k+171)c_{6,k+2} - (k+1)^7 c_{6,k} = 0$$

With the substitution

$$d_n = \frac{48^{2n}}{n!^2} c_{6,2n+1}$$

we have the recursion

$$(2n+5)(n+3)^2(n+2)^2 d_{n+3} - 32(n+2)^3(196n^2+784n+843)d_{n+2}$$

$$+64\cdot 48^2(2n+3)(14n^4+84n^3+196n^2+210n+87)d_{n+1} - 128\cdot 48^4(n+1)^5 d_n = 0$$

Consider the three solutions A_n, B_n, C_n with initial values

$$A_0 = 1, A_1 = 0, A_2 = 0$$

$$B_0 = 0, B_1 = 1, B_2 = 0$$

$$C_0 = 0, C_1 = 0, C_2 = 1$$

respectively. Let $c_{6,1} = s$, $c_{6,3} = t$. Then $c_{6,5} = \frac{5}{48} - \frac{1}{36}s + \frac{85}{72}t$ is conjectured.

Then we have

$$d_n = A_n s + 2304 B_n t + C_n(138240 - 36864s + 1566720t)$$

A related Calabi-Yau equation.

Let

$$y = \sum_{n=0}^\infty d_n x^n$$

Then y satisfies the differential equation

$$\theta^2(\theta-1)^2(2\theta-1) - 32x\theta^3(196\theta^2+59)$$

$$+64\cdot 48^2 x^2(2\theta+1)(14\theta^4+28\theta^3+28\theta^2+14\theta+3) - 128\cdot 48^4 x^3(\theta+1)^5$$

We find the mirror equation at $x=\infty$ via the substitution $\theta \longrightarrow -\theta-1$ and $x \longrightarrow 96^2 x^{-1}$

$$\theta^5 - 2x(2\theta+1)(14\theta^4+28\theta^3+28\theta^2+14\theta+3)$$

$$+4x^2(\theta+1)^3(196\theta^2+392\theta+255) - 1152x^3(\theta+1)^2(\theta+1)^2(2\theta+3)$$

This we recognize as #130 in the big table. It was found by H.Verrill [6]. The coefficients are

$$A_n = \sum_{i+j+k+l+m+s=n} (\frac{n!}{i!j!k!l!m!s!})^2$$

Seven Bessel functions.

Let

$$d_n = \frac{105^{2n}}{n!^2} c_{7,2n+1}$$

Then

$$y = \sum_{n=0}^{\infty} d_n x^n$$

satisfies

$$\theta^2(\theta-1)^2(\theta-2)^2 - 8x\theta^2(\theta-1)^2(6458\theta^2-6458\theta+2589)$$

$$+48\cdot 105^2 x^2\theta^2(658\theta^4+396\theta^2+17)$$

$$-64\cdot 105^4 x^3(84\theta^6+252\theta^5+378\theta^4+336\theta^3+180\theta^2+54\theta+7)$$

$$+256\cdot 105^6 x^4(\theta+1)^6$$

The transformation to infinity by $\theta \longrightarrow -\theta-1$ and $x \longrightarrow 210^2 x^{-1}$ gives

$$\theta^6 - x(84\theta^6+252\theta^5+378\theta^4+336\theta^3+180\theta^2+54\theta+7)$$

$$3x^2(\theta+1)^2(658\theta^4+2632\theta^3+4344\theta^2+3424\theta+1071)$$

$$-2x^3(\theta+1)^2(\theta+2)^2(6458\theta^2+19374\theta+15505)$$

$$+105^2 x^4(\theta+1)^2(\theta+2)^2(\theta+3)^2$$

with solution

$$y = \sum_{n=0}^{\infty} A_n x^n$$

where

$$A_n = \sum_{i+j+k+l+m+p+s=n} (\frac{n!}{i!j!k!l!m!p!s!})^2$$

II. Sums of squares of generalized binomial coefficients.

In [6] Verrill has given a rather complicated formula for the recursion of

$$A_n^{(k)} = \sum_{i_1+i_2+...+i_k=n} (\frac{n!}{i_1!i_2!...i_k!})^2$$

We will instead consider

$$a_n^{(k)} = \frac{A_n}{n!^2} = \sum_{i_1+i_2+...i_k=n} \frac{1}{i_1!^2 i_2!^2 ... i_k!^2}$$

Consider

$$y = \sum_{j=0}^{\infty} \frac{x^j}{j!^2}$$

Then y satisfies the differential equation

$$\theta^2 - x$$

Actually

$$y(x) = I_0(4\sqrt{x})$$

Then

$$w = y^m = \sum_{n=0}^{\infty} a_n^{(m)} x^n$$

Using Lemma 3 in J.Borwein and Salvy [5] we find the following Maple program for computing the differential equation for w for all m.

```
S:=proc(m) local M,k; M(0):=1; M(1):=t; for k to m do
M[k+1]:=x*diff(M[k],x)+M[k]*t-k*(m-k+1)*x*M[k-2]; od;
series(expand(M[m+1],x=0,infinity); end;
```

Let $m_+ = m/2$ if m is even and $m_+ = (m+1)/2$ if m is odd. Then write

$$S_m = \sum_{j=0}^{m_+} x^j Q_j(\theta)$$

Then the differential equation satisfied by

$$\sum_{n=0}^{\infty} A_n^{(m)} x^n = \sum_{n=0}^{\infty} n!^2 a_n^{(m)} x^n$$

is given by

$$\theta^{-2} \sum_{j=0}^{m_+} x^j \prod_{s=0}^{j-1} (\theta + s) Q_j(\theta)$$

III. Proof of the Main Theorem.

The Bessel function $K_0(x)$ satisfies the differential equation $T_m(x,\theta)$ given by the Maple program

$$\theta^2 - x^2$$

Using Lemma 3 in Borwein-Salvy [5] again we obtain the differential equation $T_m(x,\theta)$ satisfied by $K_0(x)^m$ given by the Maple program

```
T:=proc(m) local L,k; L(0):=1; L(1):=t; for k to m do
L[k+1]:=x*diff(L[k],x)+L[k]*t-k*(m-k+1)*x*L[k-2]; od;
series(expand(L[m+1],x=0,infinity); end;
```

The crucial part of the proof is the following

Lemma. We have

$$M_k(x,\theta) = 2^{-(k+1)} L_k(2\sqrt{x}, 2\theta)$$

Proof: We use induction on k. Assume

$$M_{k-1} = 2^{-k} L_{k-1}(2\sqrt{x}, 2\theta) \quad \text{and} \quad M_k = 2^{-(k+1)} L_k(2\sqrt{x}, 2\theta)$$

Then

$$M_{k+1} = x\frac{\partial M_k}{\partial x} + M_k\theta - xk(m-k+1)M_{k-1}$$

$$= x2^{-(k+1)}\frac{\partial}{\partial x}L_k(2\sqrt{x},2\theta)+2^{-(k+1)}L_k(2\sqrt{x},2\theta)\theta-x2^{-k}k(m-k+1)L_{k-1}(2\sqrt{x},2\theta)$$

$$= 2^{-(k+1)}x\frac{1}{\sqrt{x}}\frac{\partial}{\partial(2\sqrt{x})}L_k(2\sqrt{x},2\theta)+2^{-(k+2)}L_k(2\sqrt{x},2\theta)2\theta-(2\sqrt{x})^2 2^{-(k+2)}k(m-k+1)L_{k-1}(2\sqrt{x},2\theta)$$

$$= 2^{-(k+2)} L_{k+1}(2\sqrt{x}, 2\theta)$$

The rest of the proof is merely book-keeping. Recall that

$$T_m(x,\theta) = \sum_{j=0}^{m_+} x^{2j} P_j(\theta)$$

annihilates $K_0(x)^m$. Then by the Maple program following Example 5 in [5] we find the recursion for $c_{m,k}$ by substituting $\theta \longrightarrow -k-1-2j$ in $P_j(\theta)$. Since $k = 2n+1$ we get $\theta \longrightarrow -2(n+1+j)$ Then with

$$d_n = \frac{1}{4^n n!^2} c_{m,2n+1}$$

we get the following recursion for d_n

$$\sum_{n=0}^{m_+} n^2(n+1)^2...(n+j)^2 4^{m_+-j} P_j(-2(n+1-j))N^j = 0$$

where $Nf(n) = f(n)$. Converting to the differential equation for $y = \sum d_n x^n$ we make the substitution $n \longrightarrow \theta - j$ and $N^j \longrightarrow x^{m_+-j}$ in the coefficient of N^j

$$\sum_{j=0}^{m_+} x^{m_+-j}\theta^2(\theta-1)^2...(\theta-j)^2 4^{m_+-j} P_j(-2(\theta+1))$$

To get the differential equation at ∞ we make the substitution $\theta \longrightarrow -\theta - 1$ and $x \longrightarrow x^{-1}$ and we get

$$\sum_{j=0}^{m_+} x^j 4^j \theta^2(\theta+1)^2...(\theta+j)^2 P_j(2\theta) = \sum_{j=0}^{m_+} x^j \theta^2(\theta+1)^2...(\theta+j)^2 Q_j(\theta)$$

which is the differential equation satisfied by

$$y = \sum_{n=0}^{\infty} A_n^{(m)} x^n$$

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